International Mathematical Forum, Vol. 7, 2012, no. 30, 1483 - 1490

Action of Γ_∞ on $\mathbb{Z}$ and the Corresponding Suborbital Graphs

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Abstract

Let $\Gamma = \mathrm{PSL}\,(2, \mathbb{Z})$, the modular group. The action of Γ on the rational projective line $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ has been studied by several authors. Jones et al., [5] have shown that Γ acts transitively on $\widehat{\mathbb{Q}}$ and the stabilizer of a point is an infinite cyclic group. In this paper some properties of Γ_∞ (the stabilizer of ∞ in Γ) acting on the set $\mathbb{Z}$ of integers are investigated. It has been shown that the action is simply transitive and imprimitive. Properties of the suborbital graphs corresponding to this action have also been examined.

Mathematics Subject Classification: 05C25

Keywords: Suborbital graphs, Modular group, Group action

1. INTRODUCTION

The modular group

$$\Gamma = \mathrm{PSL}\,(2, \mathbb{Z}) = \mathrm{SL}\,(2, \mathbb{Z})\,/\,\{\pm \mathrm{I}\} \quad ,$$

is the quotient of the unimodular group SL $(2, \mathbb{Z})$ by its centre $\{\pm I\}$.

Jones et al., [5] investigated the action of Γ on the extended set of rationals, that is the rational projective line $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. The action was shown to be transitive and the stabilizer of a point an infinite cyclic group. The authors also examined the suborbital graphs for Γ on $\widehat{\mathbb{Q}}$.

Γ has a subgroup

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 (mod\ N)\right\}.$$

A lot of investigations have been done on the action of $\Gamma_o(N)$ on $\widehat{\mathbb{Q}}$ and the suborbital graphs corresponding to the action have thoroughly been investigated.

(Akbas and Baskan, [1]; Besenk et al., [2]; Güler et al., [4]; Keskin and Demirturk, [7]; Keskin [6]).

In this paper we investigate the action of Γ_∞ (the stabilizer of ∞ in Γ) on $\widehat{\mathbb{Q}}$ and the corresponding suborbital graphs.

2. THE ACTION OF Γ_∞ ON $\mathbb{Z}$

Let Γ act on $\widehat{\mathbb{Q}}$, then the stabilizer of ∞ in Γ;

$$\Gamma_\infty = \left\langle\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\rangle = \left\{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z}\right\}.$$

If we view members of $\mathbb{Z}$ to be of the form $\frac{a}{1}$, $a \in \mathbb{Z}$, then Γ_∞ acts on $\mathbb{Z}$ by

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : \frac{a}{1} \longrightarrow \frac{a+k}{1} \in \mathbb{Z}.$$

Theorem 2.1
Γ_∞ acts transitively on $\mathbb{Z}$.
Proof

Let $x, y \in \mathbb{Z}$ and $b = (y - x) \in \mathbb{Z}$.

Then $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x + b \\ 1 \end{pmatrix} = \begin{pmatrix} y \\ 1 \end{pmatrix}$. □

Theorem 2.2
The stabilizer of $0 \in \mathbb{Z}$ in Γ_∞ is the identity in Γ_∞.

Proof
Let $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$ and $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Then $\begin{pmatrix} k \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, implying $k = 0$.
Therefore stabilizer of 0 in $\mathbb{Z}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity . □
From the *Theorems 2.1* and *2.2* Γ_∞ acts regularly on $\mathbb{Z}$.

Theorem 2.3
Γ_∞ is not 2-transitive on $\mathbb{Z}$.

Proof
Put $x = 0 \in \mathbb{Z} = \Omega$ and $G = \Gamma_\infty$.
Then $G_x = 1$ (by *Theorem 2.2*).
But $(G_x, \Omega - \{x\})$ is not transitive. Hence G, Ω is not 2-transitive on Ω . □

Theorem 2.4
The action of Γ_∞ on $\mathbb{Z}$ is imprimitive.

Proof
Let $G = \Gamma_\infty$, and $x \in \mathbb{Z}$. Theorem by *Theorems 2.1* and *2.2* $G_x = 1$.
Now put $\Gamma'(n) = \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \mid k \equiv 0 (mod\ n) \right\}$.
Then $1 \leq \Gamma'(n) \leq \Gamma_\infty$ and the inclusion is strict if $n > 1$.
Therefore Γ_∞ acts imprimitively on $\mathbb{Z}$ (Wielandt, [12], P.15) . □

3. SUBORBITAL GRAPHS OF Γ_∞ ON $\mathbb{Z}$

Let (G, Ω) be a transitive permutation group. Then G acts on $\Omega \times \Omega$ by

$$g(\alpha, \beta) = \big(g(\alpha), g(\beta)\big) \quad , \quad (g \in G,\ \alpha, \beta \in \Omega).$$

The orbits of this action are called suborbits of G. The orbit containing (α, β) is denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a suborbital graph $G(\alpha, \beta)$; its vertices are the elements of Ω, and there is a directed edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$.

Clearly $O(\beta, \alpha)$ is also a suborbital, and it is either equal or disjoint from $O(\alpha, \beta)$. In the former case, $G(\alpha, \beta) = G(\beta, \alpha)$ and the graph consists of pairs of oppositely directed edges. It is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call self-paired. In the latter case, $G(\beta, \alpha)$ is just $G(\alpha, \beta)$ with arrows reversed, and we call $G(\alpha, \beta)$ and $G(\beta, \alpha)$ paired suborbital graphs.

The above ideas were first introduced by Sims [10], and are also described in a paper by Neumann [8] and in the books by Tsuzuku [11] and Biggs and White [3], the emphasis being on applications to finite groups.

If $\alpha = \beta$, then $O(\alpha, \alpha)$, is the diagonal of $\Omega \times \Omega$. The corresponding suborbital graph $G(\alpha, \alpha)$, called the trivial suborbital graph is self-paired; it consists of a loop based at each vertex $\alpha \in \Omega$. Our focus will mainly be on the remaining non-trivial suborbital graphs.

We now investigate the suborbital graphs for the action of Γ_∞ on $\mathbb{Z}$.
Since Γ_∞ acts transitively on $\mathbb{Z}$. Each suborbital contains a pair $(0, a)$ for some $a \in \mathbb{Z}$.

Also since $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : \frac{0}{1} \longrightarrow \frac{k}{1}$ and $\frac{a}{1} \longrightarrow \frac{a+k}{1}$,

$$O(0, a) = \{(k, k + a) | k \in \mathbb{Z}\}.$$

If $a = 0$, we have the trivial suborbital graph $G(0, 0)$.

Theorem 3.1
$O(0, a)$ and $O(0, b)$ are disjoint suborbitals when $a \neq b$.

Proof
$O(0, a) = O(0, b)$ if and only if a and b are in the same orbit of G_O, where $G = \Gamma_\infty$. But since $G_O = 1$, $O(0, a) = O(0, b)$ if and only if $a = b$. □

Theorem 3.2
If $a \neq 0$, the suborbital graph $G(0, a)$ is of the form shown in the diagram below

Case 1: $a > 0$

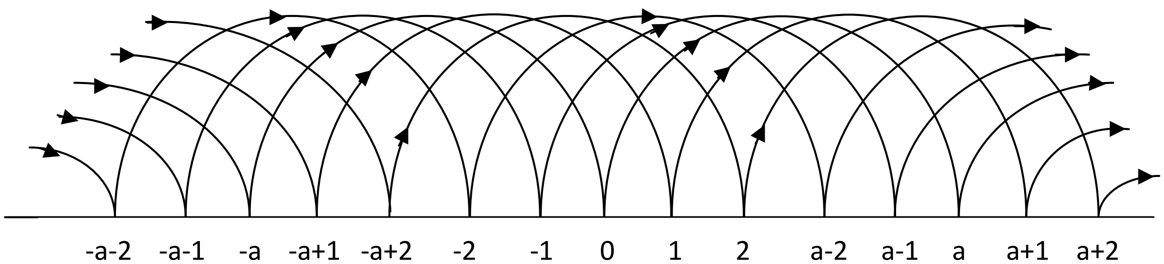

Fig 3.1

Case 1:

Case 2: $a < 0$

The graph is the same as in *Case 1* but with all directions reversed.

Proof

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} k & a+k \\ 1 & 1 \end{pmatrix}$$

So, $(0, a) \in G(o, a)$ implies $(k, a+k) \in G(o, a)$.

Case 1: If $a > 0$, then there is a directed edge from 0 to a $(0 \longrightarrow a)$ and from the calculations above, we get the given graph.

Case 2: If $a < 0$, then there is a edge from 0 to a $(a \longleftarrow 0)$ and from the calculations above we get the graph in *Case 1* with arrows reversed . □

Corollary 3.3
$G(o, a)$ is paired with $G(o, -a)$. □

Theorem 3.4
The number of connected components of $G(o, a)$ is $|a|$.

Proof
From *Theorem 3.2* it is clear that there is a path x to y $x, y \in \mathbb{Z}$ if and only if $x \equiv y \ (mod\ |a|)$. A complete residue system $(modulo\ |a|)$ has $|a|$ congruent classes; $0, 1, 2, \dots \dots, |a| - 1$. □

Corollary 3.5
$G(0, a)$ is connected if and only if $a = 1 \ or \ a = -1$.
Proof
If $|a| = 1$, then $G(0, a)$ has 1 connected component . □

Remark 3.6
If $a = 1$, then $O(0, a) = \{(k, k+1) \mid k \in Z\}$ and the corresponding graph $G(0, 1)$ is

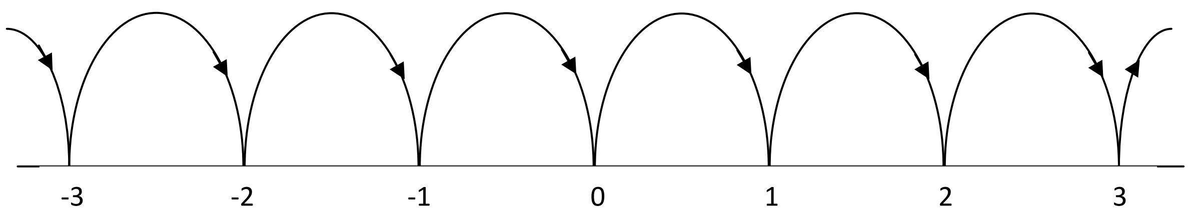

Fig 3.2(a)

If $a = -1$, then $O(0, a) = \{(k, k-1)\}$

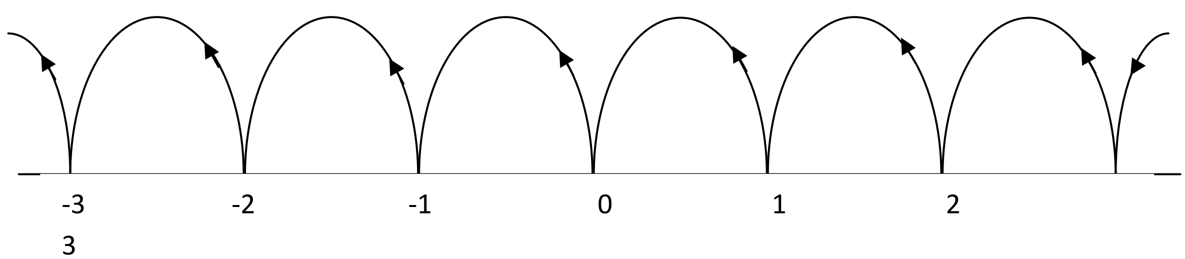

Fig 3.2(b)

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Received: December, 2011