FINITE SUBGROUPS OF THE ORTHOGONAL GROUP IN THREE DIMENSIONS AND THEIR POLES

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ABSTRACT: *The action of finite subgroups of the orthogonal group* $\mathrm{O}(\mathbb{R}^3)$ *on the set of their poles is well-known; (See Benson and Grove [1]; Neumann et al. [4]). In this paper we will use an approach different from the traditional one. We shall use the concept of marks of a permutation group. The results obtained agree with those obtained earlier (See [1], [4]).*

KEYWORDS: Orthogonal group, Poles, Group action

INTRODUCTION

An orthogonal group of a vector space *V*, denoted $\mathrm{O}(V)$, is the group of all orthogonal transformations of *V* under the binary operation of composition of maps. If $T \in \mathrm{O}(V)$, then $\det T = \pm 1$ and $T^{-1} = T^T$. The well-known finite subgroups of the orthogonal group in three dimensions are: the cyclic groups C_n; the dihedral group of degree *n*, D_n; the alternating group of degree 4, A_4; the symmetric group of degree 4, S_4 and the alternating group of degree 5, A_5. The groups D_n are isomorphic to the groups of rotations of regular *n*-gons, A_4 is isomorphic to the group of rotations of a tetrahedron, S_4 is isomorphic to the group of rotations of a cube or an octahedron and A_5 is isomorphic to the group of rotations of an icosahedron or a dodecahedron. These rotations are all done in $\mathbb{R}^3$ (3 – dimensions), otherwise in $\mathbb{R}^2$, some of them become reflections.

Notations and Preliminaries

Notation 2.1

Throughout this paper, *G* will denote a finite subgroup of the orthogonal group $O(\mathbb{R}^3)$ while τ will denote the set of poles of a finite subgroup *G* of $O(\mathbb{R}^3)$.

Definition 2.1

Let $O(\mathbb{R}^3)$ be the orthogonal group in three dimensions, then the unit sphere $S = \{p \in \mathbb{R}^3 : |p| = 1\}$ is left invariant by every transformation $T \in O(\mathbb{R}^3)$. If $T \neq 1$ (identity) is a rotation, then there are precisely two diametrically opposite points; $p, \ -p$ on the unit sphere which are left fixed by *T*. These are the points of intersection of *S* and the rotation axis for *T* and are called poles of *T*.

Theorem 2.1 (Benson and Grove [1])

Consider a finite subgroup *G* of $O(\mathbb{R}^3)$; each of its elements not equal to the identity has a pair of poles $p, \ -p$ and the set of elements of *G* with a given axis form a finite cyclic subgroup of *G*. Furthermore if τ is the set of poles of non-identity rotations of *G*, then *G* acts on τ.

Theorem 2.2 (Burnside [2])

Suppose that the number of subgroups in a finite group *G* is *s* (where a set of conjugacy class is counted once). If we collect a complete set; $G_1, G_2, \ldots, G_s$ in ascending order of their orders i.e. $|G_1| \le |G_2| \le \ldots \le |G_s|$, where G_1 = identity and $G_s = G$, then the set of corresponding coset representations; $G(/G_i), (i = 1, 2, \ldots, s)$ is the complete set of different transitive permutation representations of *G*.

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Theorem 2.3 (Burnside [2])

Any permutation representation P_G of a finite group G acting on X can be reduced into transitive coset representations with the following equation:

$$P_G = \sum \alpha_i G(/G_i) \qquad (i = 1, 2, \ldots, s), \qquad 1)$$

where the multiplicity α_i is a non-negative integer.

Definition 2.2 (Ivanov et al. [3])

The table of marks of a group G is the matrix $M(G)$, with (i, j) - entry m_{ij} equal to $m(G_j, G_i, G)$, the mark of the subgroup G_j in the coset representation $G(/G_i)$.

That is

	G_1	G_2	…	G_s
$G(/G_1)$	m_{11}	m_{12}	$\cdots$	m_{1s}
$G(/G_2)$	m_{21}	m_{22}	$\cdots$	m_{2s}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$G(/G_s)$	m_{s1}	m_{s2}	$\cdots$	m_{ss}

Theorem 2.4 (Burnside [2])

The multiplicities α_i in equation 1) are obtained by using the table of marks as;

$$\mu_j = \sum_{i=1}^{s} \alpha_i m_{ij}, \quad (j = 1, 2, \ldots, s)$$

where, μ_j is the mark of G_j in the permutation representation P_G. Furthermore if $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ is a vector with components μ_j and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ is a vector with components the multiplicities α_i in Theorem 2.3 and $M(G)$ is the table of marks of G, then

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$$\mu = \alpha M(G) \qquad 2)$$

Theorem 2.5 (Orbit – Stabilizer Theorem) (Rose [5])

Let G act on the set X, and let $x \in X$. Then

$$|Orb_G(x)| = |G : Stab_G(x)|.$$

3. Analysis of actions of Finite Subgroups of $O(R^3)$ on the set of their Poles using Table of Marks

a) Action of $G = C_n$ on τ

The cyclic group G has exactly 2 poles. The subgroups of G are of the form C_k where $k|n$ and since G is abelian each of its subgroups is normal. Suppose G has r subgroups, say $C_{1'} = 1, C_{2'}, C_{3'}, \ldots, C_{r'} = G$ with $i'|n$ and $i' \le (i+1)'$, $(i = 1, 2, \ldots, r-1)$. Then each of these subgroups fixes the 2 poles so that $\mu = (2, 2, \ldots, 2)$, an r – tuple. The table of marks of G is as shown in Table 3.1 below.

Table 3.1: Table of marks of $G = C_n$

	$C_{1'} = 1$	$C_{2'}$	$\ldots$	$C_{(r-1)'}$	$C_{r'} = G$
$G(/C_{1'})$	m_{11}				
$G(/C_{2'})$	m_{21}	m_{22}			
$\vdots$	$\vdots$	$\vdots$			
$G(/C_{(r-1)'})$	m_{r-11}	m_{r-12}	$\ldots$	m_{r-1r-1}	
$G(/C_{r'})$	1	1	$\ldots$	1	1

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, then by Equation 2) and using Table 3.1 we obtain the following system of linear equations,

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$$\alpha_1 m_{11} + \alpha_2 m_{21} + \ldots + \alpha_{r-1} m_{r-11} + \alpha_r = 2$$

$$\alpha_2 m_{22} + \ldots + \alpha_{r-1} m_{r-12} + \alpha_r = 2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\alpha_{r-1} m_{r-1r-1} + \alpha_r = 2$$

$$\alpha_r = 2$$

Since $m_{ii} \neq 0$ for all $i, \alpha_j \geq 0, \quad 1 \leq j \leq r$ and $\alpha_r = 2$, the solution to the above system of linear equations is $\alpha = (0, 0, \ldots, 0, 2)$. Hence by Theorem 2.3,

$$P_G = 2G(/G).$$

Therefore by Theorem 2.5 the action of G on τ yields 2 orbits of length one with G as the stabilizer.

b) Action of $G = D_n$ on τ

The dihedral group D_n is isomorphic to the group of rotations of a regular n – gon in three dimensions. A regular n – gon centred at the origin of $\mathbb{R}^3$ has n 2 – fold axes of rotation perpendicular to the n – fold axis. Hence $|\tau| = 2(n+1) = 2n + 2$ poles.

The subgroups of D_n depend on whether n is odd or even and are either cyclic or dihedral.

Case 1: When *n* is odd

In this case, the subgroups of D_n of order 2 lie in one conjugacy class of length n, hence its subgroups are;

i). Identity.

ii). A conjugacy class of n cyclic subgroups of order 2, C_2.

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iii). Normal cyclic subgroups $C_{m_1}, C_{m_2}, \ldots, C_{m_r}$ contained in C_n where $m_i \mid n, \quad 1 \le i \le r$.

iv). Dihedral subgroups $D_{m_1}, D_{m_2}, \ldots, D_{m_r}$ where $m_i \mid n, \quad 1 \le i \le r$.

v). A normal cyclic subgroup of order n, C_n.

vi). D_n.

The only subgroups that fix a pole are $1, C_2, C_{m_1}, \ldots, C_{m_r}$ and C_n. The identity fixes $2n+2$ poles, C_2 and C_n each fixes 2 poles and $C_{m_1}, \ldots, C_{m_r}$ each fixes 2 poles. Hence $\mu = (2n+2, 2, 2, 2, \ldots 2, 0, \ldots, 0, 2, 0)$, a $2r+4-$tuple. The corresponding table of marks of D_n (n odd) is as shown in Table 3.2.

Table 3.2: **Table of marks of $G = D_n$, n odd**

	1	C_2	C_{m_1}	$\ldots$	C_{m_r}	D_{m_1}	$\ldots$	D_{m_r}	C_n	D_n
$G(/1)$	$2n$									
$G(/C_2)$	N	1								
$G(/C_{m_1})$	m_{31}	m_{32}	m_{33}							
$\vdots$	$\vdots$	$\vdots$	$\vdots$							
$G(/C_{m_r})$	m_{r+21}	m_{r+22}	m_{r+23}	$\ldots$	m_{r+2r+2}					
$G(/D_{m_1})$	m_{r+31}	m_{r+32}	m_{r+33}	$\ldots$	m_{r+3r+2}	m_{r+3r+3}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$				
$G(/D_{m_r})$	m_{2r+21}	m_{2r+22}	m_{2r+23}	$\ldots$	$m_{2r+2r+2}$	$m_{2r+2r+3}$	$\ldots$	$m_{2r+2\,2r+2}$		
$G(/C_n)$	2	0	2	$\ldots$	2	0	$\ldots$	0	2	
$G(/G)$	1	1	1	$\ldots$	1	1	$\ldots$	1	1	1

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{r+2}, \alpha_{r+3}, \ldots, \alpha_{2r+2}, \alpha_{2r+3}, \alpha_{2r+4})$. Then by Equation 2) and using Table 3.2 we obtain,

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$$
\begin{aligned}
2n\alpha_1 + n\alpha_2 + \alpha_3 m_{31} + \ldots + \alpha_{r+2} m_{r+21} \quad + \alpha_{r+3}\, m_{r+31} \quad + \ldots + \alpha_{2r+2} m_{2r+21} \quad + 2\alpha_{2r+3} \; + \alpha_{2r+4} &= 2n+2 \\
\alpha_2 + \alpha_3 m_{32} + \ldots + \alpha_{r+2} m_{r+22} \quad + \alpha_{r+3}\, m_{r+32} \quad + \ldots + \alpha_{2r+2} m_{2r+22} \qquad\qquad + \alpha_{2r+4} &= 2 \\
\alpha_3 m_{33} + \ldots + \alpha_{r+2} m_{r+23} \quad + \alpha_{r+3}\, m_{r+33} \quad + \ldots + \alpha_{2r+2} m_{2r+23} \quad + 2_{2r+3} \quad + \alpha_{2r+4} &= 2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\alpha_{r+2} m_{r+2r+2} \quad + \alpha_{r+3} m_{r+3r+2} + \ldots + \alpha_{2r+2} m_{2r+2r+2} + 2\alpha_{2r+3} \; + \alpha_{2r+4} &= 2 \\
\alpha_{r+3} m_{r+3r+3} + \ldots + \alpha_{2r+2} m_{2r+2r+3} \qquad\qquad + \alpha_{2r+4} &= 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\alpha_{2r+2} m_{2r+22r+2} \qquad\qquad + \alpha_{2r+4} &= 0 \\
2\alpha_{2r+3} \; + \alpha_{2r+4} &= 2 \\
\alpha_{2r+4} &= 0
\end{aligned}
$$

Since $m_{ii} \neq 0$ for all i, $\alpha_j \geq 0$, $1 \leq j \leq 2r+4$ and $\alpha_{2r+4} = 0$, so $\alpha = (0,2,0,\ldots,0,0,\ldots,0,1,0)$;

by Theorem 2.3,

$$P_G = 2G(/C_2) + G(/C_n).$$

Hence by Theorem 2.5, the action of *G* on τ yields 3 orbits; 2 orbits of length *n* with C_2 as the stabilizer and 1 orbit of length 2 with C_n as the stabilizer.

Case 2: When *n* is even

In this case, the subgroups of D_n of order 2 lie in two conjugacy classes each of length $n/2$, hence its subgroups are;

i). Identity.

ii). A conjugacy class of $n/2$ cyclic subgroups of order 2 denoted by $C_2(n/2)$.

iii). A conjugacy class of $n/2$ cyclic subgroups of order 2 denoted by $C_2'(n/2)$.

iv). A normal cyclic subgroup of order 2 denoted by $C_2(1)$.

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v). Normal cyclic subgroups C_{m_1}, $C_{m_2},\ldots,C_{m_r}$ contained in C_n where $m_i \mid n$ and $m_i \neq 2,\quad 1 \le i \le r$.

vi). Dihedral subgroups D_{m_1}, $D_{m_2},\ldots,D_{m_r}$ where $m_i \mid n,\quad 1 \le i \le r$.

vii). A normal cyclic subgroup of order *n*, C_n.

viii). D_n.

The only subgroups that fix a pole are 1, $C_2\left(n/2\right)$, $C_2'\left(n/2\right)$, $C_2(1)$, $C_{m_1},\ldots,C_{m_r}$ and C_n. The identity fixes 2*n* + 2 poles, $C_2\left(n/2\right)$, $C_2'\left(n/2\right)$, $C_2(1)$ and C_n each fixes 2 poles and $C_{m_1},\ldots,C_{m_r}$ each fixes 2 poles. Hence $\mu = \left(2n+2, 2, 2, 2, 2, \ldots, 2, 0, \ldots, 0, 2, 0\right)$, a 2*r* + 6 – tuple.

The corresponding table of marks of D_n, *n* even is as shown in Table 3.3.

Table 3.3: Table of marks of *G* = *D*n, *n* even

	1	$C_2\left(n/2\right)$	$C_2'\left(n/2\right)$	$C_2(1)$	C_{m_1}	$\cdots$	C_{m_r}	D_{m_1}	$\cdots$	D_{m_r}	C_n	D_n
$G(/1)$	$2n$											
$G\left(/C_2\left(n/2\right)\right)$	n	2										
$G\left(/C_2'\left(n/2\right)\right)$	n	0	2									
$G\left(/C_2(1)\right)$	n	0	0	n								
$G\left(/C_{m_1}\right)$	m_{51}	m_{52}	m_{53}	m_{54}	m_{55}							
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$							
$G\left(/C_{m_r}\right)$	m_{r+41}	m_{r+42}	m_{r+43}	m_{r+44}	m_{r+45}	$\cdots$	m_{r+4r+4}					
$G\left(/D_{m_1}\right)$	m_{r+51}	m_{r+52}	m_{r+53}	m_{r+54}	m_{r+55}	$\cdots$	m_{r+5r+4}	m_{r+5r+5}				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$				

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$G\left(/D_{m_r}\right)$	m_{2r+41}	m_{2r+42}	m_{2r+43}	m_{2r+44}	m_{2r+45}	$\cdots$	$m_{2r+4r+4}$	$m_{2r+4r+5}$	$\cdots$	$m_{2r+42r+4}$		
$G\left(/C_n\right)$	2	0	0	2	2	…	2	0	…	0	2	
$G\left(/G\right)$	1	1	1	1	1	…	1	1	…	1	1	1

Let $\alpha = \left(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \ldots, \alpha_{r+4}, \alpha_{r+5}, \ldots, \alpha_{2r+4}, \alpha_{2r+5}, \alpha_{2r+6}\right)$. Then by Equation 2) and using Table 3.3 we obtain,

$$\begin{array}{l}
2n\alpha_1 + n\alpha_2 + n\alpha_3 + n\alpha_4 + \alpha_5 m_{51} + \ldots\ \alpha_{r+4} m_{r+41} + \alpha_{r+5} m_{r+51} + \ldots + \alpha_{2r+4} m_{2r+41} + 2\alpha_{2r+5} + \alpha_{2r+6} = 2n+2 \\
2\alpha_2 + \alpha_5 m_{52} + \ldots\ \alpha_{r+4} m_{r+42} + \alpha_{r+5} m_{r+52} + \ldots + \alpha_{2r+4} m_{2r+42} + \alpha_{2r+6} = 2 \\
2\alpha_3 + \alpha_5 m_{53} + \ldots\ \alpha_{r+4} m_{r+43} + \alpha_{r+5} m_{r+53} + \ldots + \alpha_{2r+4} m_{2r+43} + \alpha_{2r+6} = 2 \\
n\alpha_4 + \alpha_5 m_{54} + \ldots\ \alpha_{r+4} m_{r+44} + \alpha_{r+5} m_{r+54} + \ldots + \alpha_{2r+4} m_{2r+44} + 2\alpha_{2r+5} + \alpha_{2r+6} = 2 \\
\alpha_5 m_{55} + \ldots\ \alpha_{r+4} m_{r+45} + \alpha_{r+5} m_{r+55} + \ldots + \alpha_{2r+4} m_{2r+45} + 2\alpha_{2r+5} + \alpha_{2r+6} = 2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\alpha_{r+4} m_{r+4r+4} + \alpha_{r+5} m_{r+5r+4} + \ldots + \alpha_{2r+4} m_{2r+4r+4} + 2\alpha_{2r+5} + \alpha_{2r+6} = 2 \\
\alpha_{r+5} m_{r+5r+5} + \ldots + \alpha_{2r+4} m_{2r+4r+5} + \alpha_{2r+6} = 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\alpha_{2r+4} m_{2r+42r+4} + \alpha_{2r+6} = 0 \\
2\alpha_{2r+5} + \alpha_{2r+6} = 2 \\
\alpha_{2r+6} = 0
\end{array}$$

Since $m_{ii} \neq 0$ for all $i, \alpha_j \geq 0, 1 \leq j \leq 2r+6$ and $\alpha_{2r+6} = 0$, so $\alpha =$ (0,1,1,0,0,…,0,0,…,0,1,0). Therefore by Theorem 2.3,

$$P_G = G\left(/C_2\left(n/2\right)\right) + G\left(/C_2'\left(n/2\right)\right) + G\left(/C_n\right).$$

Hence by Theorem 2.5, the action of G on τ yields 3 orbits; 1 orbit of length n with $C_2\left(n/2\right)$ as the stabilizer, 1 orbit of length n with $C_2'\left(n/2\right)$ as the stabilizer and 1 orbit of length 2 with C_n as the stabilizer.

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c) Action of $G = A_4$ on τ

The alternating group A_4 is isomorphic to the group of rotations of a tetrahedron. A tetrahedron has 4 faces, 4 vertices and 6 edges, hence 7 axes of rotation. To each axis, there corresponds 2 poles, therefore $|\tau| = 2 \times 7 = 14$ poles.

Furthermore, A_4 has five conjugacy classes of subgroups. These are;

i). Identity.

ii). 3 conjugate subgroups of order 2, C_2.

iii). 4 conjugate cyclic subgroups of order 3, C_3.

iv). A normal subgroup of order 4 isomorphic to C_2 x C_2 which we shall denote by V_4.

v). A_4.

The only subgroups of A_4 that fix a pole are 1, C_2 and C_3 with 14, 2 and 2 poles fixed respectively. Thus $\mu = (14, 2, 2, 0, 0)$. The table of marks of A_4 is as shown in Table 3.4 below;

Table 3.4: Table of marks of $G = A_4$

	1	C_2	C_3	V_4	A_4
$G(/1)$	12				
$G(/C_2)$	6	2			
$G(/C_3)$	4	0	1		
$G(/V_4)$	3	3	0	3	
$G(/G)$	1	1	1	1	1

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$. Then by Equation 2) and using Table 3.4 we obtain,

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$$\begin{aligned} 12\alpha_1 + 6\alpha_2 + 4\alpha_3 + 3\alpha_4 + \alpha_5 &= 14 \\ 2\alpha_2 \qquad + 3\alpha_4 + \alpha_5 &= 2 \\ \alpha_3 \qquad + \alpha_5 &= 2 \\ 3\alpha_4 + \alpha_5 &= 0 \\ \alpha_5 &= 0 \end{aligned}$$

Thus $\alpha = (0,1,2,0,0)$. By Theorem 2.3,

$$P_G = G(/C_2) + 2G(/C_3).$$

Hence by Theorem 2.5, the action of G on τ yields 3 orbits; 1 orbit of length 6 with C_2 as the stabilizer and 2 orbits of length 4 with C_3 as the stabilizer

d) Action of $G = S_4$ on τ

The symmetric group S_4 is isomorphic to the group of rotations of a cube or an octahedron. Since a cube and an octahedron are dual polyhedra, we examine the rotational symmetries of a cube. A cube has 6 faces, 8 vertices and 12 edges, hence 13 axes of rotation.

Therefore $|\tau| = 2\times 13 = 26$ poles. Also S_4 has 11 conjugacy classes of subgroups, these are;

i). Identity.

ii). 6 conjugate subgroups of order 2 generated by permutations of the form (*ab*). A subgroup representative is denoted by C_2 (6).

iii). 3 conjugate subgroups of order 2 generated by permutations of the form (*ab*) (*cd*). A subgroup representative is denoted by C_2 (3).

iv). 4 conjugate cyclic subgroups of order 3, C_3.

v). 3 conjugate cyclic subgroups of order 4, C_4.

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vi). A normal subgroup of order 4 isomorphic to C_2 x C_2 generated by permutations of the form (*ab*) (*cd*). We denote this subgroup by $V_4(1)$.

vii). 3 conjugate subgroups of order 4 isomorphic to C_2 x C_2 generated by permutations of the form (*ab*) and (*ab*) (*cd*). A subgroup representative is denoted by V_4 (3).

viii). 4 conjugate subgroups of order 6 isomorphic to D_3.

ix). 3 conjugate subgroups of order 8 isomorphic to D_4.

x). A_4.

xi). S_4.

The only subgroups that fix a pole are $1, C_2(6), C_2(3), C_3$ and C_4 with 26, 2, 2, 2 and 2 poles fixed respectively. Thus $\mu = (26, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0)$.

The corresponding table of marks $G = S_4$ is as shown in Table 3.5 below.

Table 3.5: Table of marks of $G = S_4$

	1	$C_2(6)$	$C_2(3)$	C_3	C_4	$V_4(1)$	$V_4(3)$	D_3	D_4	A_4	S_4
$G(/1)$	24										
$G(/C_2(6))$	12	2									
$G(/C_2(3))$	12	0	4								
$G(/C_3)$	8	0	0	2							
$G(/C_4)$	6	0	2	0	2						
$G(/V_4(1))$	6	0	6	0	0	6					
$G(/V_4(3))$	6	2	2	0	0	0	2				
$G(/D_3)$	4	2	0	1	0	0	0	1			
$G(/D_4)$	3	1	3	0	1	3	1	0	1		
$G(/A_4)$	2	0	2	2	0	2	0	0	0	2	
$G(/G)$	1	1	1	1	1	1	1	1	1	1	1

Let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_{11})$, then by Equation 2) and using Table 3.5 we obtain,

$$\begin{aligned}
24\alpha_1+12\alpha_2+12\alpha_3+8\alpha_4+6\alpha_5+6\alpha_6+6\alpha_7+4\alpha_8+3\alpha_9+2\alpha_{10}+\alpha_{11}&=26\\
2\alpha_2+2\alpha_7+2\alpha_8+\alpha_9+\alpha_{11}&=2\\
4\alpha_3+2\alpha_5+6\alpha_6+2\alpha_7+3\alpha_9+2\alpha_{10}+\alpha_{11}&=2\\
2\alpha_4+\alpha_8+2\alpha_{10}+\alpha_{11}&=2\\
2\alpha_5+\alpha_9+\alpha_{11}&=2\\
6\alpha_6+3\alpha_9+2\alpha_{10}+\alpha_{11}&=0\\
2\alpha_7+\alpha_9+\alpha_{11}&=0\\
\alpha_8+\alpha_{11}&=0\\
+\alpha_9+\alpha_{11}&=0\\
2\alpha_{10}+\alpha_{11}&=0\\
\alpha_{11}&=0
\end{aligned}$$

Thus $\alpha=(0,1,0,1,1,0,0,0,0,0,0)$. By Theorem 2.3,

$$P_G=G(/C_2(6))+G(/C_3)+G(/C_4).$$

Hence by Theorem 2.5, the action of G on τ yields 3 orbits; 1 orbit of length 12 with $C_2(6)$ as the stabilizer, 1 orbit of length 8 with C_3 as the stabilizer and 1 orbit of length 6 with C_4 as the stabilizer.

e) Action of $G = A_5$ on τ

The alternating group A_5 is isomorphic to the group of rotations of an icosahedron or a dodecahedron. Since an icosahedron and a dodecahedron are dual polyhedra, we consider the rotational symmetries of an icosahedron. An icosahedron has 20 faces, 12 vertices and 30 edges, hence 31 axes of rotation. Therefore $|\tau|=2\times 31=62$ poles.

Also A_5 has 9 conjugacy classes of subgroups, these are;

i). Identity.

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ii). 15 conjugate subgroups of order 2, C_2.

iii). 10 conjugate cyclic subgroups of order 3, C_3.

iv). 5 conjugate subgroups of order 4 isomorphic to C_2 x C_2, a representative subgroup is denoted by V_4.

v). 6 conjugate cyclic subgroups of order 5, C_5.

vi). 10 conjugate subgroups of order 6 isomorphic to D_3.

vii). 6 conjugate subgroups of order 10 isomorphic to D_5.

viii). 5 conjugate subgroups of order 12 isomorphic to A_4.

ix). A_5.

The corresponding table of marks of $G = A_5$ is as shown in Table 3.6.

Table 3.6: Table of marks of $G = A_5$

	1	C_2	C_3	V_4	C_5	D_3	D_5	A_4	A_5
$G(/1)$	60								
$G(/C_2)$	30	2							
$G(/C_3)$	20	0	2						
$G(/V_4)$	15	3	0	3					
$G(/C_5)$	12	0	0	0	2				
$G(/D_3)$	10	2	1	0	0	1			
$G(/D_5)$	6	2	0	0	1	0	1		
$G(/A_4)$	5	1	2	1	0	0	0	1	
$G(/G)$	1	1	1	1	1	1	1	1	1

The only subgroups that fix a pole are $1, C_2, C_3$ and C_5 with 62, 2, 2 and 2 poles fixed respectively. Thus $\mu = (62, 2, 2, 0, 2, 0, 0, 0, 0)$. Now, let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_9)$. Then by equation 2) and using Table 3.6 we obtain,

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$$
\begin{aligned}
60\alpha_1 + 30\alpha_2 + 20\alpha_3 + 15\alpha_4 + 12\alpha_5 + 10\alpha_6 + 6\alpha_7 + 5\alpha_8 + \alpha_9 &= 62 \\
2\alpha_2 + 3\alpha_4 + 2\alpha_6 + 2\alpha_7 + \alpha_8 + \alpha_9 &= 2 \\
2\alpha_3 + \alpha_6 + 2\alpha_8 + \alpha_9 &= 2 \\
3\alpha_4 + \alpha_8 + \alpha_9 &= 0 \\
2\alpha_5 + \alpha_7 + \alpha_9 &= 2 \\
\alpha_6 + \alpha_9 &= 0 \\
\alpha_7 + \alpha_9 &= 0 \\
\alpha_8 + \alpha_9 &= 0 \\
+\alpha_9 &= 0
\end{aligned}
$$

Thus $\alpha = (0,1,1,0,1,0,0,0,0)$. By Theorem 2.3,

$$P_G = G(/C_2) + G(/C_3) + G(/C_5).$$

Hence by Theorem 2.5, the action of G on τ yields 3 orbits; 1 orbit of length 30 with C_2 as the stabilizer, 1 orbit of length 20 with C_3 as the stabilizer and 1 orbit of length 12 with C_5 as the stabilizer.

The results obtained can be summarized as shown in Table 3.7 below;

Table 3.7: Orbits and stabilizers of actions of $G \le O(\mathbb{R}^3)$ on τ

G	**|G|**	**Orbits**	$\lvert\tau\rvert$	**Order of stabilizers**		
C_n	n	2	2	n	n	
D_n	$2n$	3	$2n + 2$	2	2	N
A_4	12	3	14	2	3	3
S_4	24	3	26	2	3	4
A_5	60	3	62	2	3	5

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